Lecture 32

Measure and Ingration

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$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$T \longleftrightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

T is <u>orthogonal</u> : $\|T(\underline{x})\| = \|\underline{x}\|$.

$$\underline{A A^t = A^t A = Id.}$$

$\equiv$ $\{(a,b)\ (c,d)\}$ <u>orthogonal.</u>

$\underline{x} \in \mathbb{R}^2, \quad T(\underline{x})$

$$\cancel{\Leftarrow} \quad \|T(\underline{x})\|^2 = \langle T(\underline{x}), T(\underline{x}) \rangle$$

$$= \langle \underline{x}, T^t T(\underline{x}) \rangle$$

$$= \langle \underline{x}, \underline{x} \rangle$$

$$A^t A = Id \Rightarrow \det(A^t)\det(A) = 1$$

$$\Rightarrow (\det(A))^2 = 1$$

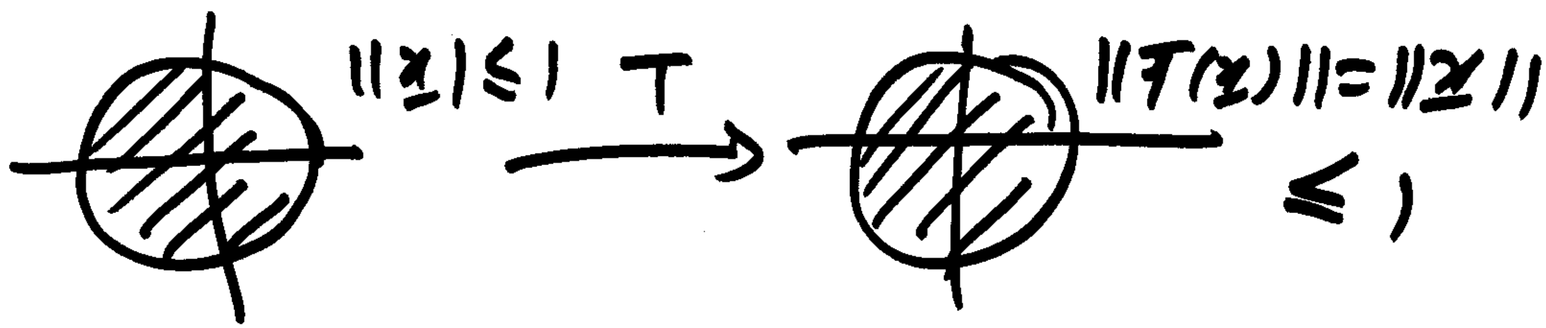
$$\Rightarrow |\det(A)| = 1.$$

T orthogonal $\Rightarrow |\det(T)| = 1$.

<u>Also</u>

$$\|T(\underline{x})\| = \|\underline{x}\|$$

$$\Rightarrow \quad \|\underline{x}\| \leq 1 \Rightarrow \|T(\underline{x})\| \leq 1$$



$$\lambda(T(|x|\le 1)) = \lambda(|x|\le 1)$$

$$= \underline{|\det(T)|}\,\lambda(|x|\le 1)$$

$$\Longrightarrow \quad c(T) = \det(T) = 1$$

T orthogonal

$$\Longrightarrow \quad c(T) = 1$$

$$T_1, T_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$T = T_1 T_2$$

~~Pf~~ let $E \in \mathcal{B}_{\mathbb{R}^2}$

$$\lambda_{\mathbb{R}^2}(T_1 T_2(E)) = C(T_1 T_2)\,\lambda_{\mathbb{R}^2}(E)$$

$$\|$$

$$\lambda_{\mathbb{R}^2}(T_1(T_2(E)))$$

$$\|$$

$$C(T_1)\,\lambda_{\mathbb{R}^2}(T_2(E)) = C(T_2)\,C(T_1)\,\lambda_{\mathbb{R}^2}(E)$$

$$\Rightarrow \quad C(T_1 T_2) = C(T_1)\,C(T_2)$$

$N \subseteq \mathbb{R}^2$, $\lambda^*_{\mathbb{R}^2}(N) = 0$.

$\forall \varepsilon > 0$, $\exists$ rectangles $\{R_i\}_{i \geq 1}$

such that $N \subseteq \bigcup_{i=1}^{\infty} R_i$ — $\circledast$

and $\sum_{i=1}^{\infty} \lambda_{\mathbb{R}^2}(R_i) < \varepsilon$

Each $R_i \in \mathcal{B}_{\mathbb{R}^2}$, $\Rightarrow$ $T(R_i) \in \mathcal{B}_{\mathbb{R}^2}$

$$\lambda_{\mathbb{R}^2}(T(R_i)) = |\det(T)| \, \lambda_{\mathbb{R}^2}(R_i)$$

$\circledast \Rightarrow$ $T(N) \subseteq \bigcup_{i=1}^{\infty} T(R_i)$

$$\Rightarrow \lambda^*_{\mathbb{R}^2}(T(N)) \leq \sum_{i=1}^{\infty} \lambda_{\mathbb{R}^2}(T(R_i))$$

$$= |\det(T)| \sum_{i=1}^{\infty} \lambda_{\mathbb{R}^2}(R_i)|$$

$$\leq |\det(T)|\, \varepsilon$$

$\forall \varepsilon > 0$, let $\varepsilon \longrightarrow 0$

$$\Rightarrow \lambda^*_{\mathbb{R}^2}(T(N)) = 0, \text{ if}$$

T is nonsingular

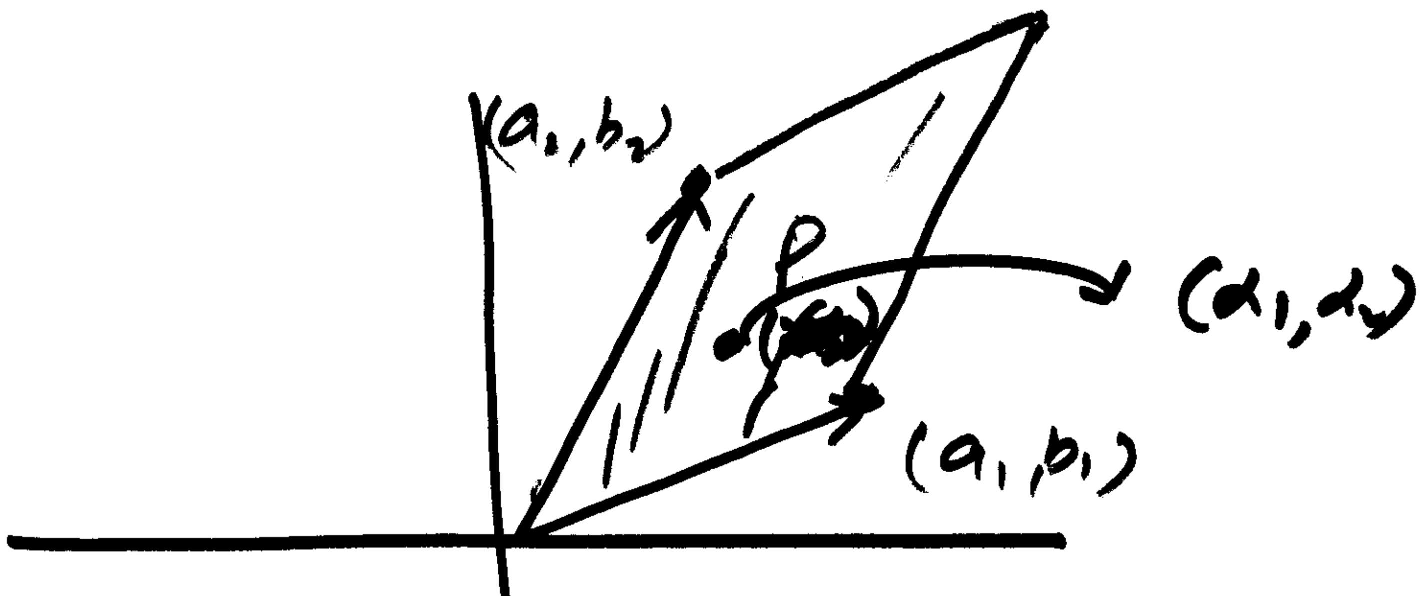
Let $A \in \mathcal{L}_{\mathbb{R}^2}$

Then $A = E \cup N, \quad E \in \mathcal{B}_{\mathbb{R}^2}$

and $\lambda_{\mathbb{R}^2}(A) = \lambda_{\mathbb{R}^2}(E)$

$$\underline{T(A)} = \underline{T(E)} \cup \underline{T(N)}$$

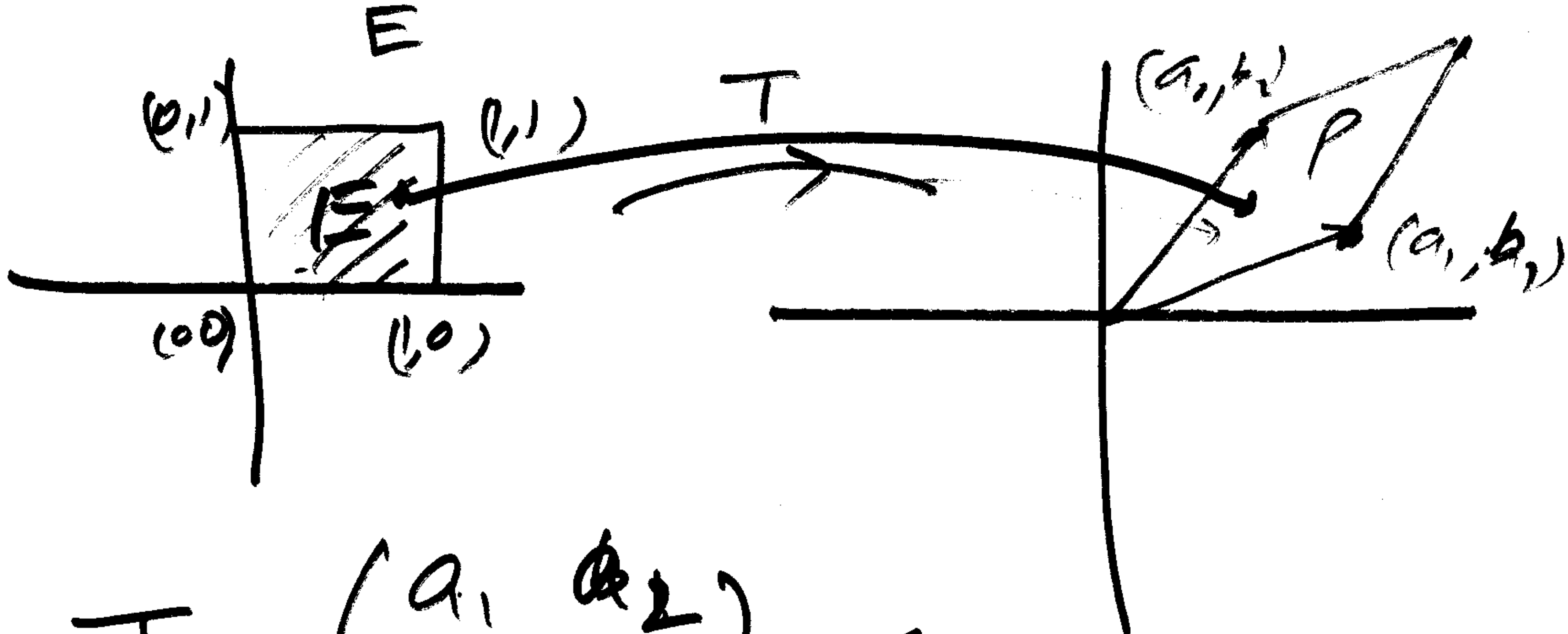
$$\begin{aligned}\lambda_{\mathbb{R}^2}(T(A)) &= \lambda_{\mathbb{R}^2} T(E) \\ &= |\det(T)|\,\lambda_{\mathbb{R}^2}(E) \\ &= |\det(T)|\,\lambda_{\mathbb{R}^2}(A)\end{aligned}$$



$$P = \{(\alpha_1 a_1 + \alpha_2 a_2,\ \alpha_1 b_1 + \alpha_2 b_2) \mid \begin{array}{l} 0 \le \alpha_1 \le 1 \\ 0 \le \alpha_2 \le 1 \end{array}$$

$$\lambda(P) = |a_1 b_2 - a_2 b_1|$$

$P = T(E)$, T — linear
E — a nice set $\|$

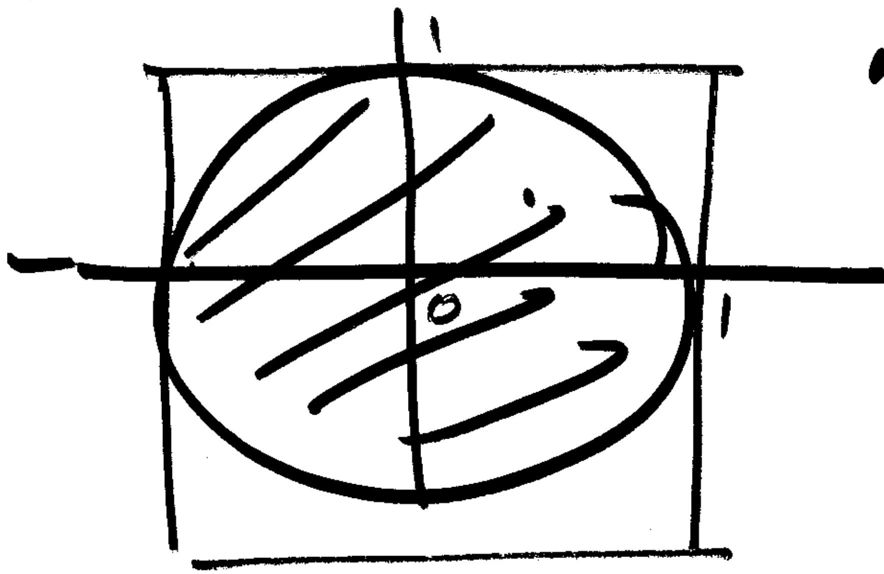


$$T = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \checkmark$$

$$T(E) = P$$

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \underline{\begin{pmatrix} a_1\alpha_1 + a_2\alpha_2 \\ b_1\alpha_1 + b_2\alpha_2 \end{pmatrix}}$$

$$\lambda_{\mathbb{R}^2}(P) = \lambda_{\mathbb{R}^2}(T(E)) = |\underline{\det T}|\,\underline{\lambda_{\mathbb{R}^2}(E)}$$

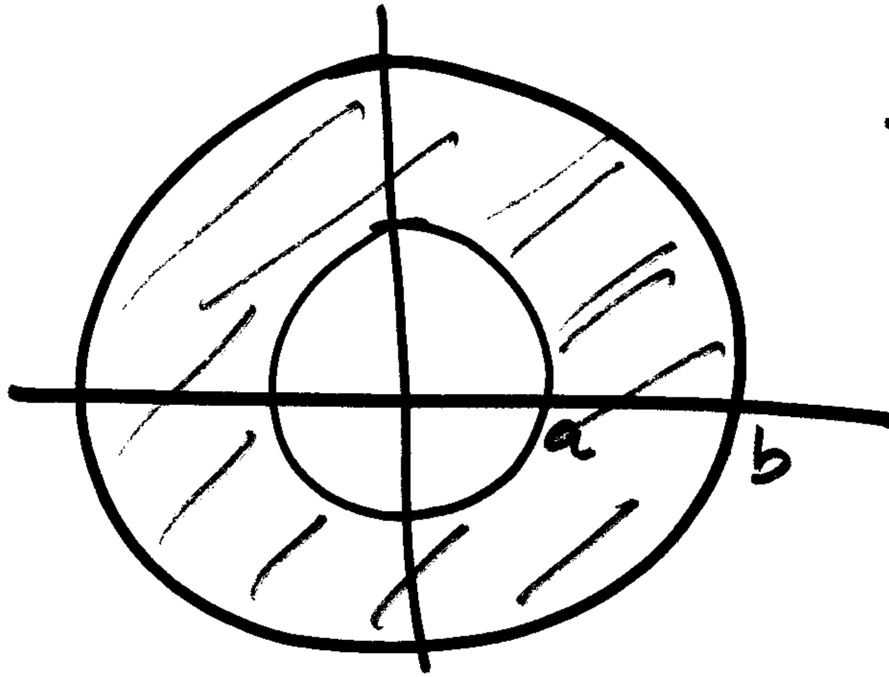


$\{x^2+y^2<1\} \subseteq S$ is a bounded set

$$\boxed{\lambda_{\mathbb{R}^2}(x^2+y^2<1)} \leq \lambda_{\mathbb{R}^2}(S) < +\infty$$

$:= \pi$

1



$\pi(b^2 - a^2)$

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ ay \end{pmatrix}$$

$x^2+y^2<1$

$(ax)^2+(ay^2)$
$= a^2(x^2+y^2$
$< a^2$

$$E = \{(x,y) \mid x^2+y^2<1\}$$

$$T(E) = \{(x,y) \mid x^2+y^2<a^2\}$$

$$\underline{\lambda_{\mathbb{R}^2}(T(E))} = |\det(T)|\, \lambda_{\mathbb{R}^2}(E)$$

$$= \underline{a^2\, \pi}$$

$$\lambda\{(x,y) \mid x^2+y^2<a^2\} = \pi a^2$$

$$\lambda_{\mathbb{R}^2}(\{(x,y) \mid x^2+y^2<b^2\}) < +\infty$$

$$\lambda_{\mathbb{R}^2}(a^2 < x^2+y^2 < b^2)$$

$$= \lambda_{\mathbb{R}^2}\left(\{x^2+y^2<b^2\} \setminus \{x^2+y^2<a^2\}\right)$$

$$= \lambda_{\mathbb{R}^2}(\{x^2+y^2<b^2\}) - \lambda_{\mathbb{R}^2}(\{x^2+y^2<a^2\})$$

$$= \pi b^2 - \pi a^2$$

$$= \underline{\pi(b^2-a^2)}$$